GENERAL CRITERION OF FRACTURE OF CRACKED PLATES

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Analogous to finding the stress state corresponding to equilibrium of a system in continuum mechanics, we state a general criterion of fracture which is an evident modification of the Griffith theory. We shall show that Rice's criterion is a particular case of the one proposed.

We first recall the theorem of minimum complementary energy: among all the statically admissible stress fields (i.e., those satisfying equilibrium equations in volume V, and the condition of prescribed tractions on the part S_σ of the boundary S), the real one, corresponding to compatibility of the system minimises the functional

$$G^* = \int_V U\,dV - \int_V F_i u_i\,dV - \int_{S_u} T_i u_i\,dS$$

where $U = \int_0^\sigma \varepsilon\,d\sigma$, F_i = body forces, T_i = tractions, u_i = displacements and S_u, the complement of S_σ ($S_u + S_\sigma$ = whole boundary). G^* is called the complementary energy of the system. In the particular case where $F_i=0$ and the tractions are prescribed over the entire boundary ($S_u=0$) of a linear elastic medium

$$G^* = G = 1/2 \int_V \sigma_{ij}\,\varepsilon_{ij}\,dV = \text{elastic strain energy}$$

Considering a correspondence between the continuous and the fractured medium and suitably interpreting the latter case, the state of stability (being the one where the crack becomes just critical), and the variable involved (being the crack length), Griffith proposed a fracture criterion based on minimum of new potential energy which takes into account also the energy of the formation of the crack. Thus Griffith's fundamental equation governing the instability of a cracked plate is usually found in the literature as

$$\frac{\partial W_o}{\partial a} - \frac{\partial W_1}{\partial a} = \frac{\partial S}{\partial a}$$

where W_o = work done by external forces,

W_1 = stored strain energy = $\int_V W\,dV$

$W = \int_0^\varepsilon \sigma\,d\varepsilon$

and S represents the surface energy of the crack. The equation becomes in the particular case of a linear elastic cracked medium:

$$\partial G_g/\partial a = 0 \qquad \text{with } \partial^2 G_g/\partial a^2 \geq 0$$

where $G_g = G - S$

which was verified on brittle materials like glass.

In general, however, $G^* \neq G$. Moreover, in the formation of the crack, apart from surface energy, there are other irreversible effects such as plastic energy dissipation, as pointed out by Irwin [2]. An evident modification of the Griffith criterion for its general application would therefore be to replace G by G^* and S by Γ in Griffith's functional G_g, where Γ represents the sum of all energies associated with the formation of the crack. We have, therefore, the following theorem:

Among all cracks satisfying the condition of traction free surfaces, the equilibrium crack is given by the minimum of

$$G_p = G^* - \Gamma$$

In case body forces are neglected,

$$G_p = \int_V U\,dV - \int_{S_u} T_i u_i\,dS - \Gamma$$

The Rice [3] integral is (see Fig. 1)

$$J = \int_\gamma [W dy - T_i\,\partial u_i/\partial x\,dt]$$

where dt is an element of arc length, is such that the hatched area representing energy on the force-displacement curve (Fig. 2) for two specimens, one having a crack length a, the other a+Δa, is given by

$$W_R = J\Delta a$$

under fixed-load or fixed-grip conditions. Therefore,

$$J\Delta a = G^*\big|_{a+\Delta a} - G^*\big|_a = \Delta G^*$$

and the fracture criterion of Rice implies

$$\partial G^*/\partial a = J_c = \text{a constant at rupture.}$$

This represents a particular case of the proposed criterion, i.e., the case where energy required Γ is given by

$$\Gamma = c_o\,a + c_1\,G^*$$

where c_o and c_1 are suitable constants.

REFERENCES

[1] A. A. Griffith, *Philosophical Transactions of the Royal Society* A 221 (1921) 163-198.

[2] G. R. Irwin, *Fracturing of Metals*, American Society for Metals, Cleveland, (1948) 147-166.

[3] J. R. Rice, in *Fracture-an Advanced Treatise*, H. Liebowitz, ed., Academic Press, 2 (1968) 191-311.

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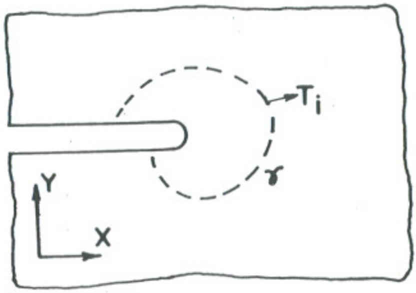

FIG. 1

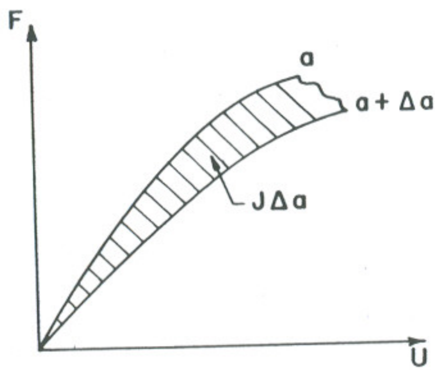

FIG. 2